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The generalized asymmetric dynamic covariance model

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Abstract

In this paper we extend the ADC model of Kroner and Ng [1998. Review of Financial Studies 11, 817–844] such that it allows for cross-asymmetries in conditional volatility. That is, the model allows for asymmetries in covariances after shocks of opposite signs. We find evidence for significant cross-asymmetries in the conditional volatility in stock and bond markets.

Keywords: Stock and bond market interaction; Time-varying covariances; Asymmetric volatility

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1. Introduction

While there exists a large amount of literature on time-varying conditional variances of either stock and bond returns, the number of studies on conditional covariances between these returns is rather limited. One of the most influential studies on modeling time-varying covariances is a study by Kroner and Ng (1998). They introduced the Asymmetric Dynamic Covariance (ADC) model, a multivariate GARCH model that nests several other multivariate models and allows for asymmetric volatility. However, their approach does

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not take into account cross-asymmetric volatilities: the conditional variance and covariance between asset returns can be higher (or lower) after a negative shock in one asset and a positive shock in the other asset, rather than shocks of opposite signs of the same magnitude. Recently, De Goeij and Marquering (2004) showed that cross-asymmetries can be statistically and economically significant in a multivariate GJR (Glosten et al., 1993) setting. As we concentrate in this note on the interaction between stocks and bonds, we observe many shocks of opposite signs. We propose an extension of the ADC model that incorporates cross-asymmetries in conditional variances and covariances. We refer to this model as the Generalized Asymmetric Dynamic Covariance (GADC) model. To test the appropriateness of the model we examine the asymmetric volatility behavior of stock and bond market returns.

2. The generalized asymmetric dynamic covariance model

Consider the multivariate return equation

$$r_{t+1} = \mu_{t+1} + \varepsilon_{t+1}, \tag{1}$$

where $\mu_{t+1} = E\{r_{t+1} \mid \mathcal{I}_t\}$ is the conditional mean and $\mathcal{I}_t$ denotes the information known at time t. The error term ε_{t+1} has the properties

$$E\{\varepsilon_{t+1} \mid \mathcal{I}_t\} = 0, \qquad E\left\{\varepsilon_{t+1}^2 \mid \mathcal{I}_t\right\} = H_{t+1},$$

where H_{t+1} is the conditional covariance matrix, consisting of conditional covariances $(h_{ij,t+1})$ and conditional variances $(h_{ii,t+1})$.

Next, define ε_{t-1}^{+} as a vector with elements $\varepsilon_{i,t-1}^{+} = \max[0, \varepsilon_{i,t-1}]$ and ε_{t-1}^{-} as a vector with elements $\varepsilon_{i,t-1}^{-} = \min[0, \varepsilon_{i,t-1}]$. We extend the ADC model by adding symmetric transformations of $(\varepsilon_{t-1}^{+}\varepsilon_{t-1}^{-\prime})$ and $(\varepsilon_{t-1}^{-}\varepsilon_{t-1}^{+\prime})$. The operator $\Xi(.)$, makes a non-symmetric matrix symmetric, using the elements of the lower-triangle part of the matrix. That is

$$\Xi \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}. \tag{2}$$

The GADC model becomes:

$$H_t = D_t R D_t + \Phi \odot \Theta_t, \tag{3}$$

where

$$H_t = (h_{ij,t}), \tag{4}$$

$$D_t = (d_{ij,t}), \quad d_{ii,t} = \sqrt{\theta_{ii,t}}, \quad \forall i, \quad \text{and} \quad d_{ij,t} = 0, \quad \forall i \neq j, \tag{5}$$

$$\Theta_t = (\theta_{ij,t}), \tag{6}$$

$$R = (\rho_{ij}), \quad \rho_{ii} = 1, \quad \forall i, \quad \text{and} \quad \rho_{ij} = \rho_{ji}, \tag{7}$$

$$\Phi = (\phi_{ij}), \quad \phi_{ii} = 0, \quad \forall i, \quad \text{and} \quad \phi_{ij} = \phi_{ji} \quad \text{and} \tag{8}$$

$$\begin{aligned}\theta_{ij,t} = {} & \omega_{ij} + b_i' H_{t-1} b_j + a_i' \varepsilon_{t-1} \varepsilon_{t-1}' a_j + g_i' \varepsilon^-_{t-1} \varepsilon^{-\prime}_{t-1} g_j \\ & + k_{1,i}' \Xi\bigl(\varepsilon^+_{t-1} \varepsilon^{-\prime}_{t-1}\bigr) k_{1,j} + k_{2,i}' \Xi\bigl(\varepsilon^-_{t-1} \varepsilon^{+\prime}_{t-1}\bigr) k_{2,j}, \quad \forall i, j.\end{aligned} \tag{9}$$

The vectors b_i, a_i, g_i, $k_{1,i}$ and $k_{2,i}$, $i = 1, \ldots, N$, are $N \times 1$ vectors of parameters, and R and $\Omega = (\omega_{ij})$ are positive definite and symmetric matrices. Without additional restrictions, not all coefficients are identified. It can be shown that the system of equations resulting from (9) do not give us sufficient information to identify all parameters, and consequently to estimate the parameters $\kappa_{l,11}$, $\kappa_{l,12}$, $\kappa_{l,21}$ and $\kappa_{l,22}$, $l = 1, 2$, uniquely. To guarantee identification of the cross-asymmetry parameters, it is assumed that $\kappa_{l,ij} = \kappa_{l,ji}$ for $l = 1, 2$. The assumption of a symmetric $K_l = (\kappa_{l,ij,t})$ matrix seems the most natural choice, as Kroner and Ng (1998) resolve a similar identification problem for the ADC model by assuming that the matrix $B = (b_{ij})$ is symmetric, i.e., $b_{ij} = b_{ji}$.[1] Note that the ADC model of Kroner and Ng (1998) is obtained when $\kappa_{1,ij} = \kappa_{2,ij} = 0$, $\forall i, j$. The two cross-asymmetric terms account for a different effect for opposite shocks in asset returns in addition to the existing negative return shock effects of the ADC model.

As indicated in the introduction, many multivariate models are nested in the ADC model. Consequently, it is interesting to examine the effect of the extension on the following popular multivariate models: VECH, CCORR, BEKK and FARCH. More details on these models and the conditions under which the ADC reduces to other multivariate models are given in Kroner and Ng (1998). Besides the conditions stated in Kroner and Ng (1998), the GADC model reduces to the asymmetric VECH model under the additional condition $k_{1,i} = \kappa_{1,i}\iota_i$ and $k_{2,i} = \kappa_{2,i}\iota_i$, $\forall i$, where ι_i is the ith column of an $N \times N$ identity matrix:

Asymmetric VECH:

$$h_{ii,t} = \omega_{ij} + \beta_i^2 h_{ii,t-1} + \alpha_i^2 \varepsilon^2_{i,t-1} + \gamma_i^2 \bigl(\varepsilon^-_{i,t-1}\bigr)^2, \quad \forall i, \tag{10}$$

$$\begin{aligned}h_{ij,t} = {} & \phi_{ij}\omega_{ij} + \phi_{ij}\beta_i\beta_j h_{ij,t-1} + \phi_{ij}\alpha_i\alpha_j \varepsilon_{i,t-1}\varepsilon_{j,t-1} + \phi_{ij}\gamma_i\gamma_j \varepsilon^-_{i,t-1}\varepsilon^-_{j,t-1} \\ & + \phi_{ij}\kappa_{1,i}\kappa_{1,j}\varepsilon^+_{i,t-1}\varepsilon^-_{j,t-1} + \phi_{ij}\kappa_{2,i}\kappa_{2,j}\varepsilon^-_{i,t-1}\varepsilon^+_{j,t-1}, \quad \forall i \neq j.\end{aligned} \tag{11}$$

The asymmetric CCORR, BEKK and FARCH do not require additional conditions:

Asymmetric CCORR:

$$h_{ii,t} = \omega_{ij} + \beta_i^2 h_{ii,t-1} + \alpha_i^2 \varepsilon^2_{i,t-1} + \gamma_i^2 \bigl(\varepsilon^-_{i,t-1}\bigr)^2, \quad \forall i, \tag{12}$$

$$h_{ij,t} = \rho_{ij}\sqrt{h_{ii,t}}\sqrt{h_{jj,t}}, \quad \forall i \neq j. \tag{13}$$

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[1] Moreover, an alternative strategy by restricting one of the parameters to be zero would have the unwanted effect that important asymmetric terms become zero.

Asymmetric BEKK:

$$H_t = \Omega + B' H_{t-1} B + A' \varepsilon_{t-1} \varepsilon'_{t-1} A + G' \varepsilon^-_{t-1} \varepsilon^{-\prime}_{t-1} G \\ + K'_1 \Xi \left(\varepsilon^+_{t-1} \varepsilon^{-\prime}_{t-1}\right) K_1 + K'_2 \Xi \left(\varepsilon^-_{t-1} \varepsilon^{+\prime}_{t-1}\right) K_2, \tag{14}$$

where $A = [a_1, \ldots, a_N]$, $B = [b_1, \ldots, b_N]$, $G = [g_1, \ldots, g_N]$, $K_1 = [k_{1,1}, \ldots, k_{1,N}]$ and $K_2 = [k_{2,1}, \ldots, k_{2,N}]$.

Asymmetric Factor ARCH (FARCH):

$$h_{ij,t} = \sigma_{ij} + \lambda_i \lambda_j h_{p,t}, \quad \forall i, j, \tag{15}$$

$$h_{p,t} = \omega_p + \beta h_{p,t-1} + \alpha \varepsilon^2_{p,t-1} + \gamma \left(\varepsilon^-_{p,t-1}\right)^2, \quad \forall i \neq j, \tag{16}$$

where $h_{p,t} = w' H_t w$, $\varepsilon_{pt} = w' \varepsilon_t$, $\varepsilon^-_{p,t} = w' \varepsilon^-_t$, and $\sigma_{ij} = \omega_{ij} - \lambda_i \lambda_j w' \Omega w$, with λ_i and λ_j are parameter scalars. The FARCH model assumes that there is a single portfolio, with weights denoted by vector w, with a variance driving all the conditional covariances in (15). This factor follows a univariate GARCH model, see (16). Thus, cross-asymmetries, which are only non-zero in multivariate settings,[2] cannot be incorporated in univariate models. Note that (10) and (12) correspond to the Glosten et al. (1993) univariate asymmetry model. The asymmetric VECH specification in (10),(11) is, after reparameterizing, exactly the one used in De Goeij and Marquering (2004). The asymmetric CCORR allows for asymmetric variance, but not (by construction) for asymmetries in covariances. The BEKK model on the other hand allows very naturally for cross asymmetries, as the BEKK model is most similar to the more general ADC model. We conclude that from the four popular multivariate models above, only the VECH and BEKK models reduce to a specification in which cross-asymmetries are accounted for. The other two, the asymmetric CCORR and FARCH models, are exactly as in Kroner and Ng (1998). Consequently, when cross-asymmetries are important, it is not advisable to employ a CCORR or FARCH specification.

3. Empirical results

To investigate the cross-asymmetries in stock and bond markets, the ADC and GADC models are estimated using daily returns on the S&P 500 index and the 10 year US Treasury bond. In a first estimation step, a simple VAR model for Eq. (1) is estimated[3] and its residuals are used in a second step to estimate the ADC and GADC models using quasi-maximum likelihood (QML).[4]

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[2] Note that the product $\varepsilon^+_i \varepsilon^-_i$ is always equal to zero.

[3] We used SIC and BIC criteria to determine the optimal amount of lags in the VAR system. The estimates of the resulting VAR(5) model show some significant relations between the S&P 500 returns and lagged S&P 500 and lagged 10 year Treasury bond returns. The same holds for the 10 year Treasury bond return.

[4] A two-step procedure employing QML provide consistent and asymptotically normal estimators. Moreover, in a recent paper, Andreou and Werker (2004) show that for certain types of first-stage models, including our VAR model, applying QML to estimate GARCH models in a second step does not results in a loss of efficiency. More

Table 1

Estimation results: Conditional covariance between the S&P 500 index and 10 year Treasury bond

	ADC model		GADC model	
	Estimate	St. err.	Estimate	St. err.
ρ_{12}	0.0011	0.0062	0.0134	0.0101
ϕ_{12}	1.0037^*	0.0085	0.9831^*	0.0156
$\omega_{11}(\times 1000)$	14.0278	8.4599	12.5672	7.5184
$\omega_{12}(\times 1000)$	1.7698	1.8071	1.4717	2.0866
$\omega_{22}(\times 1000)$	4.2622^*	1.1924	4.2363^*	1.2239
b_{11}	0.9581^*	0.0156	0.9614^*	0.0151
b_{12}	0.0033	0.0024	0.0020	0.0027
b_{22}	0.9645^*	0.0059	0.9653^*	0.0059
a_{11}	0.1923^*	0.0321	0.1916^*	0.0332
a_{21}	0.0002	0.0111	0.0027	0.0096
a_{12}	0.0010	0.0250	0.0040	0.0348
a_{22}	0.2049^*	0.0155	0.2022^*	0.0174
g_{11}	-0.2471^*	0.0707	-0.2261^*	0.0767
g_{21}	0.0402^*	0.0132	0.0376^*	0.0132
g_{12}	-0.0196	0.0458	-0.0289	0.0658
g_{22}	-0.0957^*	0.0355	-0.0872	0.0550
$k_{1,11}$	.	.	-0.2115^*	0.1055
$k_{1,12}$	.	.	0.0731	0.0879
$k_{1,22}$	.	.	0.0243	0.0256
$k_{2,11}$	.	.	0.4345	0.3740
$k_{2,12}$	.	.	-0.1096^*	0.0358
$k_{2,22}$	.	.	0.0144	0.0187
Log likelihood	-9093.55		-9077.70	

Notes. Index 1 refers to the S&P 500 index, whereas index 2 refers to the 10 year Treasury bond. There are 4898 observations used in the estimation. Robust Bollerslev and Wooldridge (1992) standard errors are presented.

[*] Statistically significant at the 5% level.

The estimation results are presented in Table 1. The parameters ϕ_{12} and ρ_{12} indicate to what extent the variability over time of the covariance between the two assets is caused by the assets' individual variances. The model implies that for $\phi_{12} = 0$ and $\rho_{12} = 1$ the conditional covariance is completely determined by the conditional variances of both assets. We can see from the table that the estimated coefficients for ρ_{12} are very close but not significantly different from zero for both specifications. In addition, the estimated coefficients for ϕ_{12} are not significantly different from one, while these are significantly different from zero. Consequently, time variability and asymmetric effects in the conditional covariance are not due to the occurrence of these effects in the conditional variances.

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specifically, in an example, Andreou and Werker (2004) show that for the estimation of a GARCH model using QML, employing the residuals of a first stage ARMA model estimated by least squares, the limiting distribution of the residuals is the same as that applied to the innovations themselves. A necessary condition for this result is that the score function of the first step estimator is asymptotically uncorrelated with the second step estimator, which in our case is fulfilled.

The b_{11} and b_{22} parameters are around 0.96 and highly significant. These parameter values indicate that the conditional variances for stock and bonds are highly persistent. The a_{ij}-parameters measure the persistency of returns shocks and the estimates are positive for both models. The g_{ij}-parameters measure the persistency of negative return shocks. For the ADC as well as for the GADC specifications most of these parameter estimates are statistically significant from zero at the five percent level. The estimates imply that, in contrast to bond returns, conditional stock market variance responds asymmetrically to stock and bond market shocks. In addition, the GADC model predicts that the conditional covariance is relatively low after negative stock return shocks and positive bond return shocks, rather than shocks of opposite sign. In sum, the estimates provide empirical evidence for the asymmetric effects in conditional (co)variances.

To check the validity of the models we perform some specification tests. When testing the models against each other, we find that the GADC model is statistically superior to the ADC model. The corresponding LR test statistic, following a χ_6^2-distribution, is 31.60, with a p-value of 0.00. Thus we find statistical evidence for asymmetry in the conditional (co)variances of stock and bond market returns.

Next, we compare the ex-post cross-product matrix of the vector of residuals to the estimated covariance matrices by measuring the vertical distance between $\varepsilon_{i,t}\varepsilon_{j,t}$ and $h_{ij,t}$. Define the generalized residual as $u_{ij,t} = \varepsilon_{i,t}\varepsilon_{j,t} - h_{ij,t}$. If the model is correct, $E_{t-1}\{u_{ij,t}\} = 0$. In addition, $u_{ij,t}$ should be uncorrelated with any variable known at time $t-1$. Therefore, a natural way to identify misspecification is to examine whether $u_{ij,t}$ is correlated with variables known at time $t-1$.[5] Because the main feature of the GADC model is asymmetry, we define the following sign indicators:

$$\begin{aligned}
MI_{1,t-1} &= I(\varepsilon_{1,t-1} < 0),\\
MI_{2,t-1} &= I(\varepsilon_{2,t-1} < 0),\\
MI_{3,t-1} &= I(\varepsilon_{1,t-1} < 0;\, \varepsilon_{2,t-1} < 0),\\
MI_{4,t-1} &= I(\varepsilon_{1,t-1} < 0;\, \varepsilon_{2,t-1} > 0),\\
MI_{5,t-1} &= I(\varepsilon_{1,t-1} > 0;\, \varepsilon_{2,t-1} < 0),\\
MI_{6,t-1} &= I(\varepsilon_{1,t-1} > 0;\, \varepsilon_{2,t-1} > 0).
\end{aligned}$$

These indicators allow to test for several types of asymmetries in the data.[6] To complete the testing design, we use the robust conditional moment test framework of Wooldridge (1990). The test statistic is defined as

$$TS_{rcm} = \left(\frac{1}{T}\sum_{t=1}^{T} u_{ij,t}\lambda_{g,t-1}\right)^2 \left(\frac{1}{T}\sum_{t=1}^{T} u_{ij,t}^2\lambda_{g,t-1}^2\right)^{-1}, \tag{17}$$

where $\lambda_{g,t-1}$ is the residual from a regression of the misspecification indicator $MI_{g,t-1}$, $g = 1, \ldots, 6$, on the derivatives of $h_{ij,t}$ with respect to the parameters of the model. Un-

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[5] See Brenner et al. (1996) for a detailed description of this type of tests.

[6] In addition, as pointed out by for example Engle and Ng (1993), the magnitude of the shocks can play an important role. Corresponding, untabulated, tests showed that both models do not exhibit misspecification in this respect.

Table 2
Diagnostic tests for covariance specification

	ADC	GADC
Panel A: Robust conditional moment tests		
MI_1	7.786*	5.379*
MI_2	0.231	0.153
MI_3	1.273	1.641
MI_4	4.731*	1.042
MI_5	5.188*	0.637
MI_6	2.048	2.447
Panel B: Ljung–Box tests for serial correlation in $\varepsilon_{1t}\varepsilon_{2t}/h_{12t}$		
$Q(6)$	2.562	11.80
$Q(12)$	2.654	17.37
$Q(18)$	2.691	17.76
$Q(24)$	22.280	20.08

Notes. Panel A presents the robust conditional moment test statistics. The misspecification indicators are listed in the first column. The statistic is χ^2_1-distributed. Panel B presents the Ljung–Box test statistic for serial correlation in the standardized cross-product of residuals. $Q(r)$ is the Ljung–Box statistic for rth order serial correlation. The 95% critical values for $Q(6)$, $Q(12)$, $Q(18)$ and $Q(24)$ are 12.6, 21.0, 28.9 and 36.4, respectively.

* Significance at the 5 % level.

der some regularity conditions, Wooldridge (1990) shows that TS_{rcm} has an asymptotic χ^2_1 distribution. As can be seen from Panel A of Table 2 the GADC model does a better job in capturing the asymmetries in the covariances than the ADC. Especially the cross-asymmetry tests MI_4 and MI_5 show that the GADC is capable of explaining existing asymmetries whereas the ADC model fails. Finally, the Ljung–Box tests in Panel B of Table 2 show that there is no serial correlation left in $\varepsilon_{1t}\varepsilon_{2t}/h_{12t}$.

4. Concluding remarks

The empirical results show that the GADC model better fits the data than the ADC model. We show that there are cross-asymmetric effects in the conditional variances and covariance of stock and bond returns. In contrast to the bond market variance, the model predicts that the conditional stock market variance responds asymmetrically to stock and bond market shocks. The GADC model predicts that the conditional covariance is relatively low after negative stock return shocks and positive bond return shocks rather than shocks of opposite signs. We also find that the conditional correlation between stock and bond returns varies significantly over time. An investor that holds a diversified asset portfolio should therefore rebalance his asset holdings over time accordingly.

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